ICOSAHOM 2014

ON FORMULATIONS OF DISCONTINUOUS GALERKIN AND FLUX RECONSTRUCTION METHODS FOR CONSERVATION LAWS

H. T. Huynh /win/
NASA Glenn Research Center
Cleveland, Ohio, USA

High-Order Methods

Discontinuous Galerkin (DG) methods by Reed and Hill 1973, Cockburn and Shu 1990’s, Bassi and Rebay 1997, 2000 …

- Integral form, stable, powerful machinery
- Not intuitive

Staggered-Grid methods by Kopriva and Kolias1996; Spectral Difference (SD) scheme by Liu, Vinokur, and Wang 2004, ...

- Differential form, simple and intuitive
- Mildly unstable

Flux Reconstruction methods (FR, Huynh 2007, Wang and Gao 2009, Jameson 2010, Vincent, Castonguay, Jameson 2011, ...)

- Differential form, recovers DG, SD, Spectral Volume
- Simple, economical, and intuitive
- Stability proofs (Jameson 2010, Vincent el al. 2011,…)

Outline

- Review DG method
- New strong form (approximate delta functions)
- FR methods by integrating the new strong form
- Fourier and energy stability
- Conclusions

Conservation Laws

Conservation law

$$u_t + f_x = 0$$

with initial condition

$$u(x,0) = u_{\text{init}}(x).$$

Calculate the solution $u(x,t)$

Legendre Polynomials

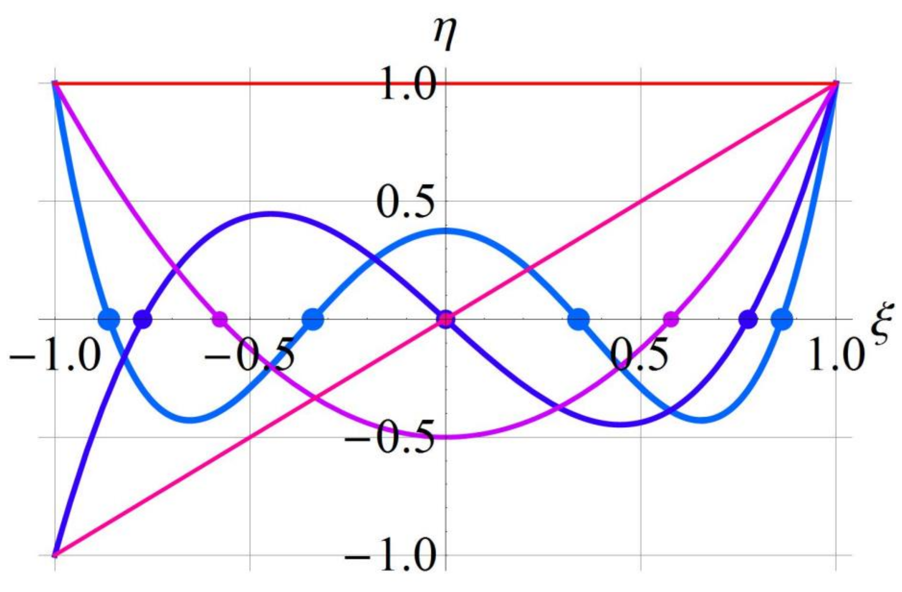
Let $\boldsymbol{P}_m$ be the space of polynomials of degree m or less.

On $I = [-1,1]$, for any two continuous functions v and w

$$(v,w)_I = (v,w) = \int_{-1}^{1} v(\xi)w(\xi)d\xi$$

Let the Legendre polynomial of degree i be denoted by L_i and defined by

$$L_i \perp \boldsymbol{P}_{i-1} \quad \text{and} \quad L_i(1) = 1.$$

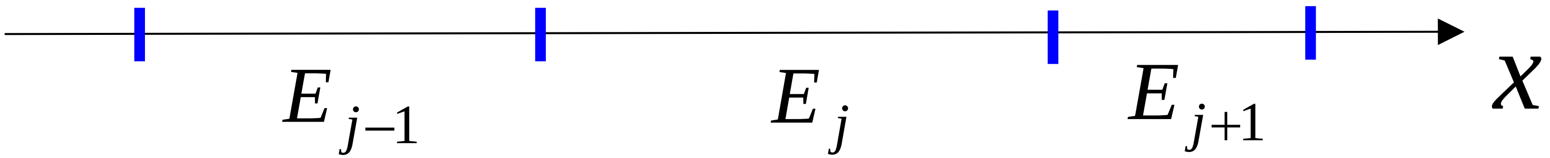


Projection

On $I = [-1,1]$, the projection of a function v onto $\boldsymbol{P}_m$ is

$$\mathscr{P}_m(v) = \sum_{i=0}^{m} \frac{(v, L_i)}{(L_i, L_i)} L_i .$$

Discretization



For each cell E_j, with the local coordinate ξ on $[-1,1]$,

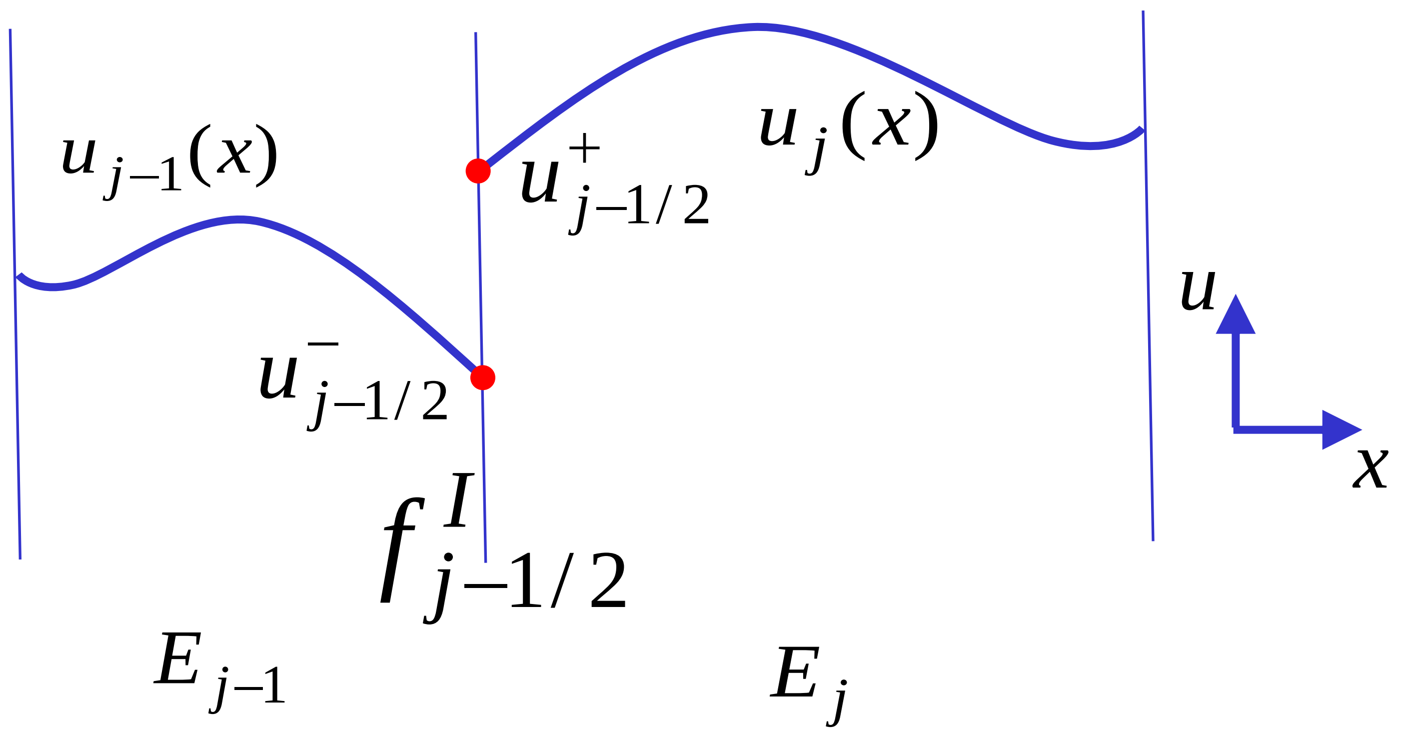
$$u_j(\xi) = \sum_{i=0}^{k} u_{j,i} L_i(\xi)$$

At time t^n, (dropping superscript n) suppose the data $u_{j,i}$ are known for all j and i.

We wish to calculate f_x for $(u_j)_t + (f(u_j))_x = 0.$

Interface Flux

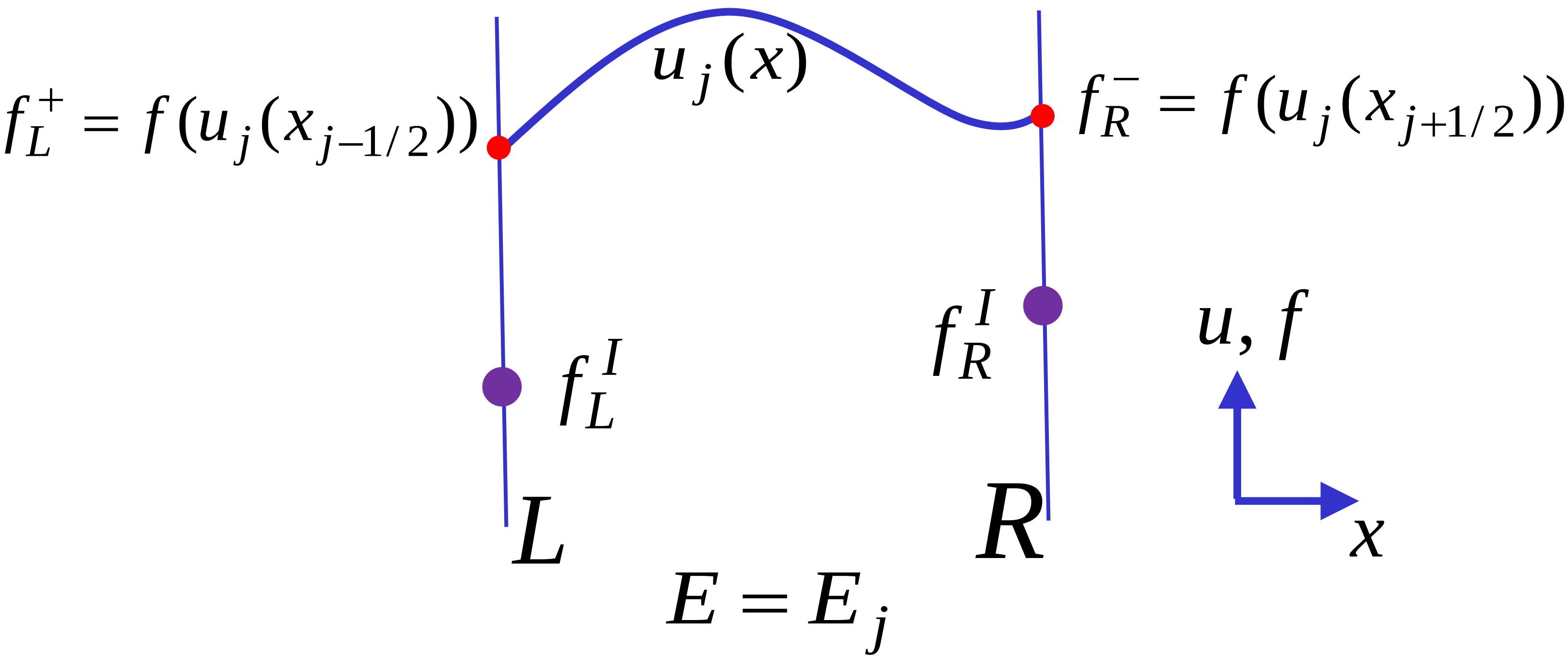
At each interface $j-1/2$, using $u^-_{j-1/2}$ and $u^+_{j-1/2}$, define a flux $f^I_{j-1/2}$ (say, Roe's flux) common for the two adjacent cells



Jumps at interfaces

On $E = E_j$, denote $(u,v)_E = \int_E u(x)v(x)dx.$

Set $[f]_L = f_L^I - f_L^+$ and $[f]_R = f_R^I - f_R^-$.



Review DG Formulation

On E, with test function ϕ (degree k),

$$(u_h, \phi)_t + ((f(u_h))_x, \phi) = 0.$$

Integrate by parts,

$$(u_h, \phi)_t + (f\phi)_{\partial E} - (f(u_h), \phi_x) = 0.$$

Allow data across cells to interact by

$$(u_h, \phi)_t + (f^I \phi)_{\partial E} - (f(u_h), \phi_x) = 0.$$

The above is the weak form. Equivalently,

$$(u_h, \phi)_t + f_R^I \phi_R - f_L^I \phi_L - (f(u_h), \phi_x) = 0.$$

Review DG Formulation

Weak form: on E

$$(u_h,\phi)_t + f_R^I\phi_R - f_L^I\phi_L - (f(u_h),\phi_x) = 0.$$

With $[f]_L = f_L^I - f_L^+$ and $[f]_R = f_R^I - f_R^-$,

integrate by parts again, we obtain the strong form

$$(u_h,\phi)_t + ((f(u_h))_x,\phi) + [f]_R\phi_R - [f]_L\phi_L = 0.$$

The task is to eliminate ϕ.

Approximate Dirac Delta Function

* For a fixed α on $I=[-1,1]$, let the approximate (Dirac) delta function to degree k at α be a linear functional on $\boldsymbol{P}_k$:

$$\delta_\alpha(\phi)=\phi(\alpha).$$

* There exists a polynomial of degree k denoted by $\gamma_{\alpha,k}=\gamma_\alpha$, i.e., $\gamma_\alpha \in \boldsymbol{P}_k$, such that

$$(\gamma_\alpha,\phi)=\phi(\alpha).$$

* Proof. Set $\gamma_\alpha=\sum_{i=0}^k b_i L_i$. Then $(\gamma_\alpha, L_m)=(\sum_{i=0}^k b_i L_i, L_m)$, or $L_m(\alpha)=b_m(L_m,L_m)$, or $b_m=L_m(\alpha)(2m+1)/2$.

  That is,

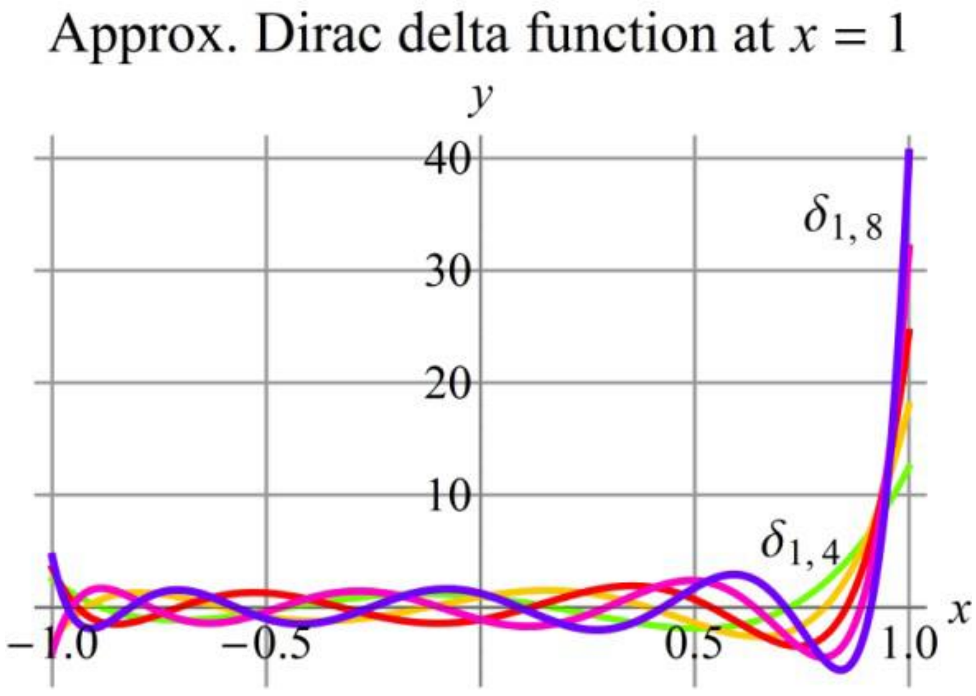
$$\gamma_\alpha=\delta_\alpha=\delta_{\alpha,k}=\sum_{i=0}^k \frac{2i+1}{2} L_i(\alpha) L_i.$$

Approximate Dirac Delta Function

$$\delta_{-1,k} = \sum_{i=0}^{k} \frac{2i+1}{2}(-1)^i L_i \quad \text{and} \quad \delta_{1,k} = \sum_{i=0}^{k} \frac{2i+1}{2} L_i .$$

$$\|L_i\| = \sqrt{\tfrac{2}{2i+1}}$$

$$\left\|\tfrac{2i+1}{2} L_i\right\| = \sqrt{\tfrac{2i+1}{2}}$$



New Strong Form

Standard strong form

$$(u_h,\phi)_t + ((f(u_h))_x,\phi) + [f]_R\phi_R - [f]_L\phi_L = 0.$$

Using the approximate delta functions,

$$(u_h,\phi)_t + ((f(u_h))_x,\phi) + [f]_R(\delta_R,\phi) - [f]_L(\delta_L,\phi) = 0.$$

Using the projection onto $\boldsymbol{P}_k$,

$$(u_h,\phi)_t + (\mathscr{P}_k([f(u_h)]_x),\phi) + [f]_R(\delta_R,\phi) - [f]_L(\delta_L,\phi) = 0.$$

New strong form

$$(u_h)_t + \mathscr{P}_k\big((f(u_h))_x\big) + [f]_R\delta_R - [f]_L\delta_L = 0.$$

Three Members of a Family of FR Schemes

Scheme DG

$$(u_h)_t + \mathscr{P}_k\big((f(u_h))_x\big) +$$

$$[f]_R\left(\delta_{R,k-1} + \frac{2k+1}{2}L_k\right) - [f]_L\left(\delta_{L,k-1} + (-1)^k\,\frac{2k+1}{2}L_k\right) = 0.$$

Scheme g_{Ga}

$$(u_h)_t + \mathscr{P}_k\big((f(u_h))_x\big) +$$

$$[f]_R\left(\delta_{R,k-1} + \frac{k+1}{2}L_k\right) - [f]_L\left(\delta_{L,k-1} + (-1)^k\,\frac{k+1}{2}L_k\right) = 0.$$

Scheme g_2

$$(u_h)_t + \mathscr{P}_k\big((f(u_h))_x\big) +$$

$$[f]_R\left(\delta_{R,k-1} + \frac{k}{2}L_k\right) - [f]_L\left(\delta_{L,k-1} + (-1)^k\,\frac{k}{2}L_k\right) = 0.$$

New Strong Forms

Strong form S1

$$(u_h)_t + \mathscr{P}_k\big((f(u_h))_x\big) + [f]_R\,\delta_R - [f]_L\,\delta_L = 0.$$

Strong form S2

$$(u_h)_t + \big(\mathscr{P}_k(f(u_h))\big)_x + [f]_R\,\delta_R - [f]_L\,\delta_L = 0.$$

- Derivative with no interaction: projection or interpolation;

  for form S1, interpolate via chain rule: $(f(u))_x = a(u)\, u_x$
- Interaction: approximate delta function, exact to degree k.

Energy-Stable FR (ESFR) Schemes

Strong form S1 and S2 for DG method (linear advection),

$$(u_h)_t + a(u_h)_\xi + [f]_R\left(\delta_{R,k-1} + \tfrac{2k+1}{2} L_k\right)$$
$$- [f]_L\left(\delta_{L,k-1} + (-1)^k \tfrac{(2k+1)}{2} L_k\right) = 0.$$

ESFR schemes made simple: $\alpha_k > 0$

$$(u_h)_t + a(u_h)_\xi + [f]_R(\delta_{R,k-1} + \alpha_k L_k)$$
$$- [f]_L\left(\delta_{L,k-1} + (-1)^k \alpha_k L_k\right) = 0.$$

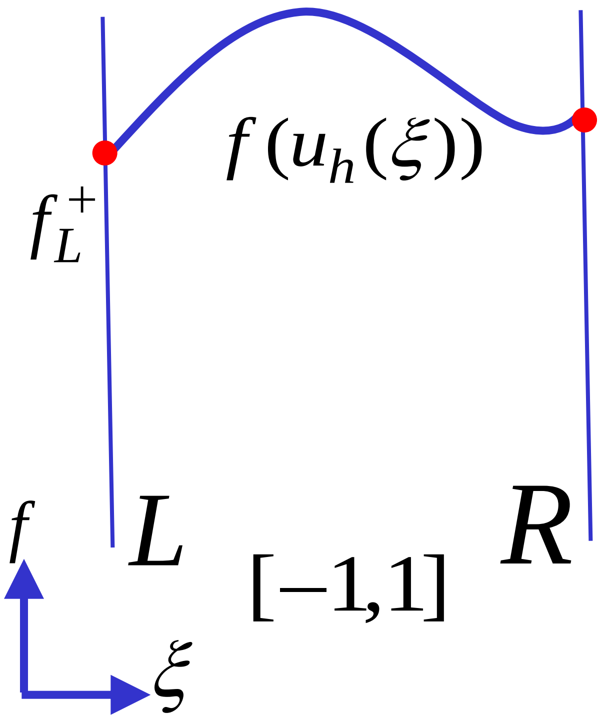
Key idea of the proof : Differentiate k times in ξ

$$\left(\frac{d^k u_h}{d\xi^k}\right)_t + [f]_R\left(\alpha_k \frac{d^k L_k}{d\xi^k}\right) - [f]_L\left((-1)^k \alpha_k \frac{d^k L_k}{d\xi^k}\right) = 0.$$

Reconstructing the Flux by Integrating the Strong Form S1

S1 $$(u_h)_t + \mathscr{P}_k\big((f(u_h))_x\big) + [f]_R\,\delta_R - [f]_L\,\delta_L = 0.$$

1. Flux polynomial (no interaction), i.e., discontinuous flux function, deg. $k+1$



$$f_{\text{IPD}}(\eta) = f_L^+ + \int_{-1}^{\eta} \mathscr{P}_k\big((f(u_h))_\xi\big)\,d\xi$$

f_{IPD} of degree $k+1$ determined by

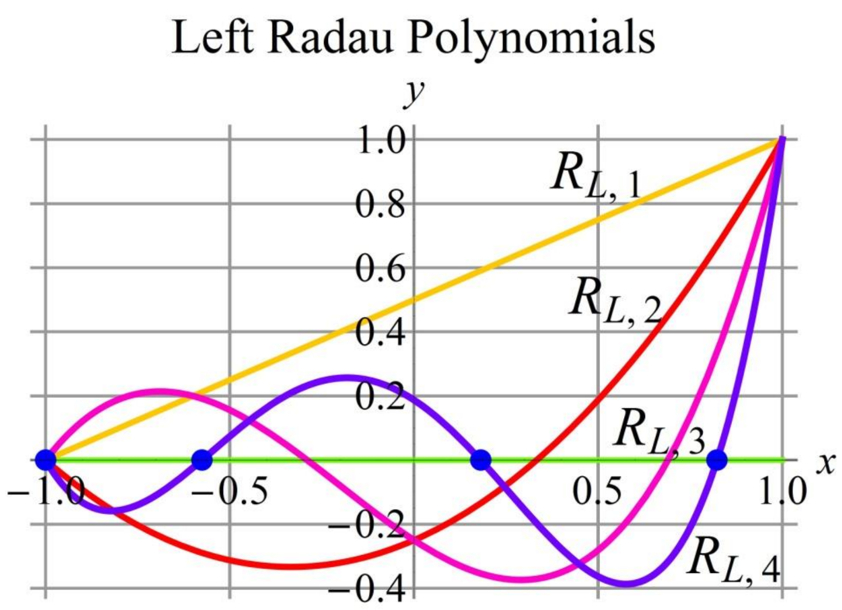
$$f_{\text{IPD}}(-1) = f_L^+, \quad f_{\text{IPD}}(1) = f_R^-$$

and $\mathscr{P}_{k-1}(f_{\text{IPD}}) = \mathscr{P}_{k-1}(f(u_h))$

FR: Integrate the Strong Form S1

S1 $$(u_h)_t + \mathscr{P}_k\left(\left(f(u_h)\right)_x\right) + [f]_R\delta_R - [f]_L\delta_L = 0.$$

2(a). Correction function for the right boundary



$$g_R(\xi) = \int_{-1}^{\xi} \delta_R(\eta)\, d\eta$$

$$g_R' = \delta_R$$

g_R is of degree $k+1$:

$$g_R(-1) = 0,$$

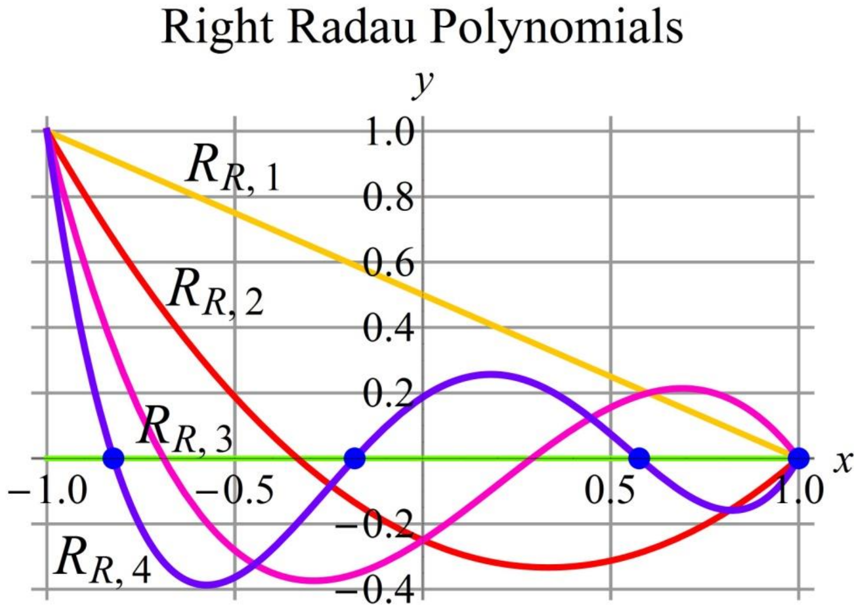
$$g_R(1) = 1,$$

$$\mathscr{P}_{k-1}(g_R) = 0.$$

FR: Integrate the Strong Form S1

S1 $$(u_h)_t + \mathscr{P}_k\big((f(u_h))_x\big) + [f]_R\delta_R - [f]_L\delta_L = 0.$$

2(b). Correction function for the left boundary



$$g_L(\xi) = \int_\xi^1 \delta_L(\eta)\,d\eta$$

$$g_L' = -\delta_L$$

g_L is of degree $k+1$:

$$g_L(-1) = 1,$$

$$g_L(1) = 0,$$

$$\mathscr{P}_{k-1}(g_L) = 0.$$

Flux Reconstruction Form

On E, for nonlinear conservation laws, set

$$F = f_{\mathrm{IPD}} + [\,f\,]_L \, g_L + [\,f\,]_R \, g_R \, .$$

Then F is of degree $k+1$ determined by

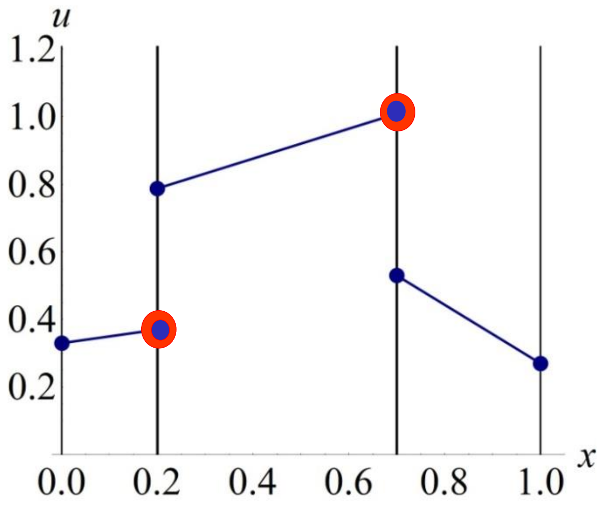
$$F(-1) = f_L^I, \qquad F(1) = f_R^I,$$

and $$\mathscr{P}_{k-1}(F) = \mathscr{P}_{k-1}(\,f(u_h)).$$

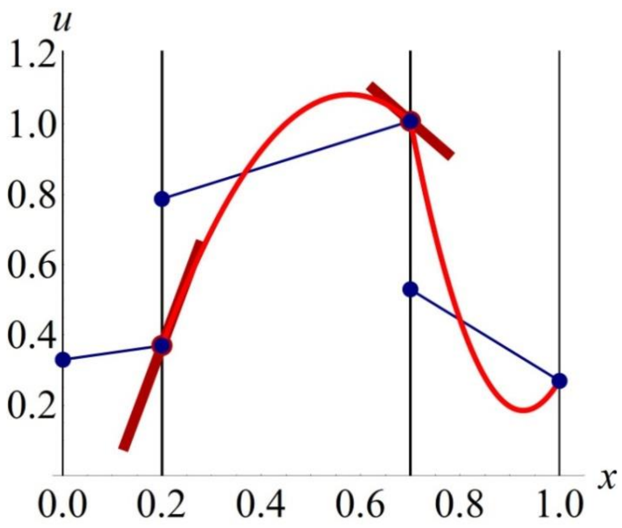
Also, $$F_\xi = \mathscr{P}_k\Big((\,f(u_h))_\xi\Big) + [\,f\,]_L \, \delta_L + [\,f\,]_R \, \delta_R \, .$$

Reconstructing the Flux

Example: advection equation with $k = 1$.



(a) Data



(b) DG

A Family of Fourier Stable FR Schemes

Let g_L of deg. $k+1$ be defined by

$$g_L(-1) = 1, \quad g_L(1) = 0,$$

and k additional conditions.

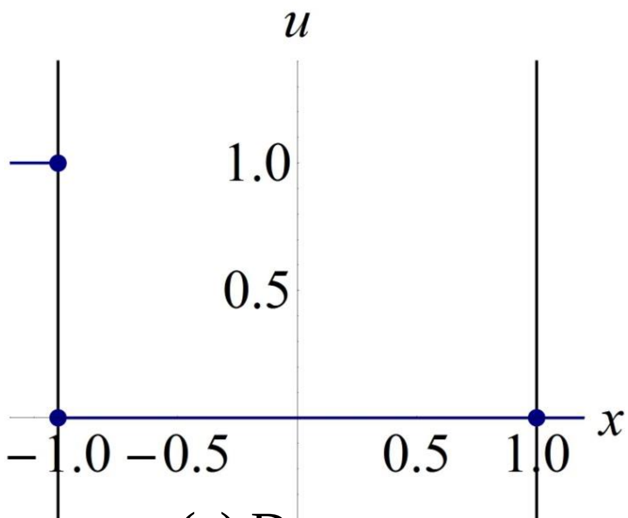
For DG,

$$\mathscr{P}_{k-1}(g_L) = 0.$$

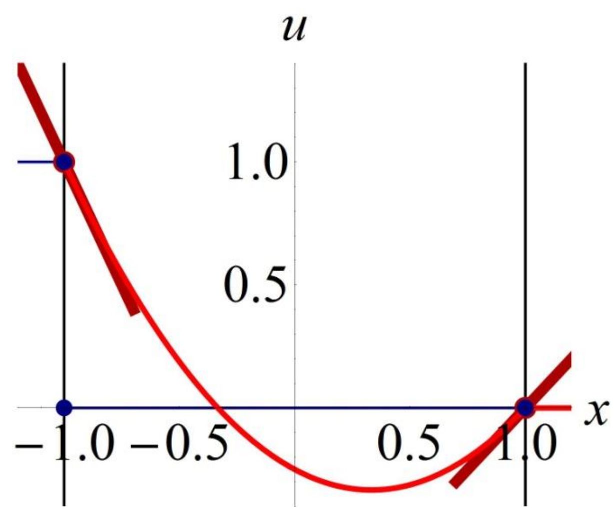
For a family of stable schemes,

$$\mathscr{P}_{k-2}(g_L) = 0.$$

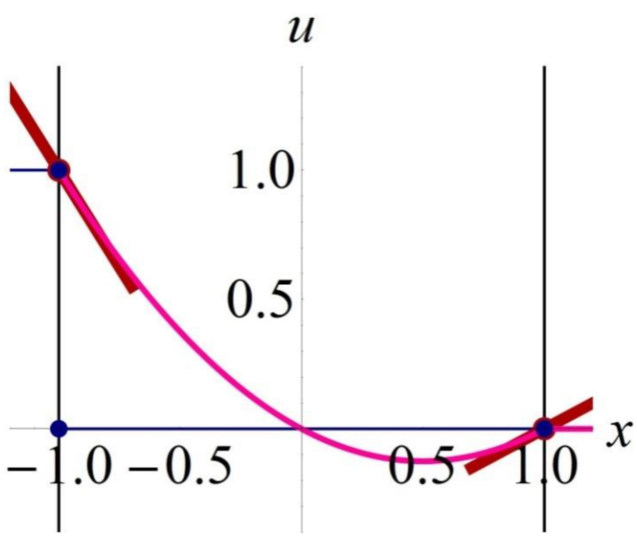
Correction functions ($k = 1$)



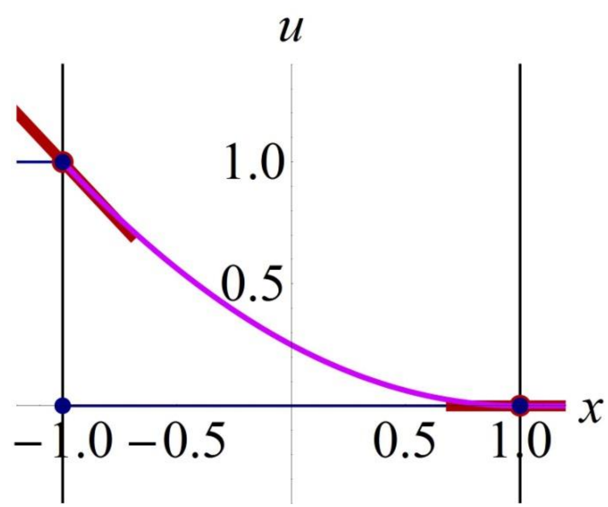
(a) Data



(b) DG (projection)

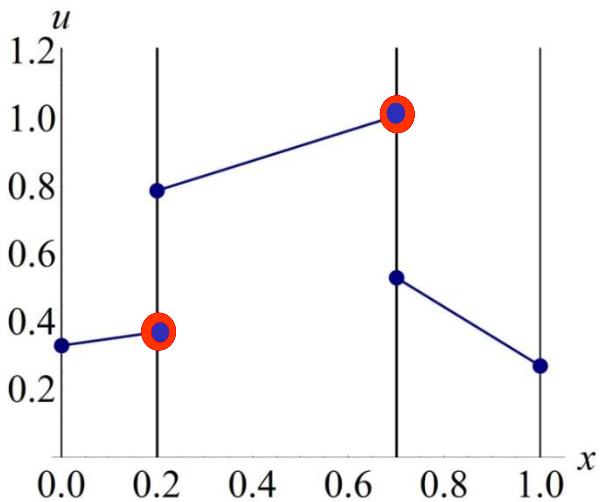


(c) Scheme g_{Ga} (SD)

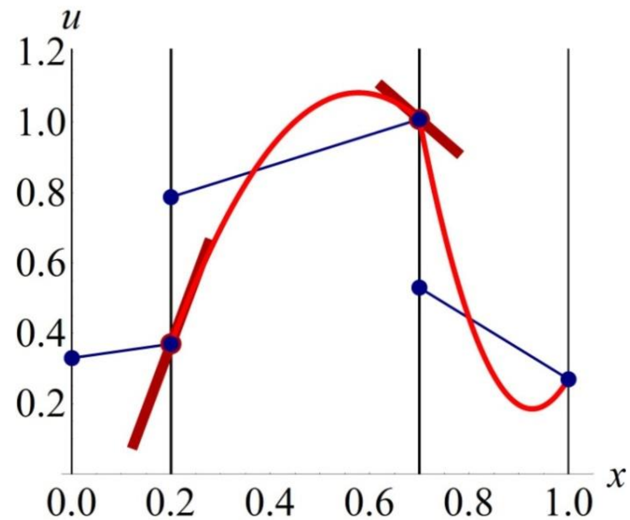


(d) Scheme g_2

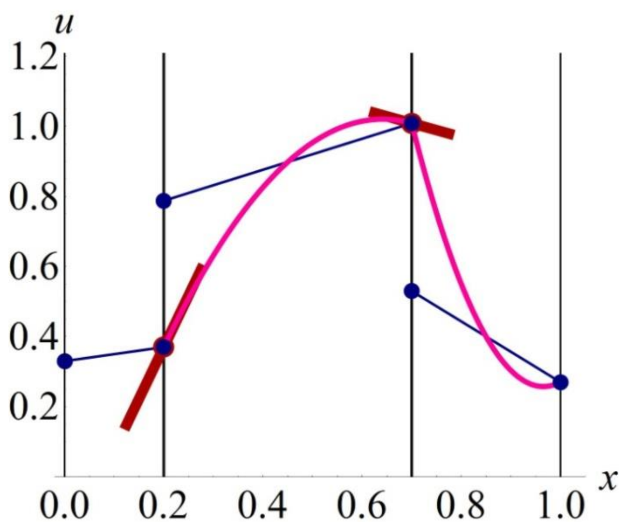
Second-Order FR Schemes ($k = 1$)



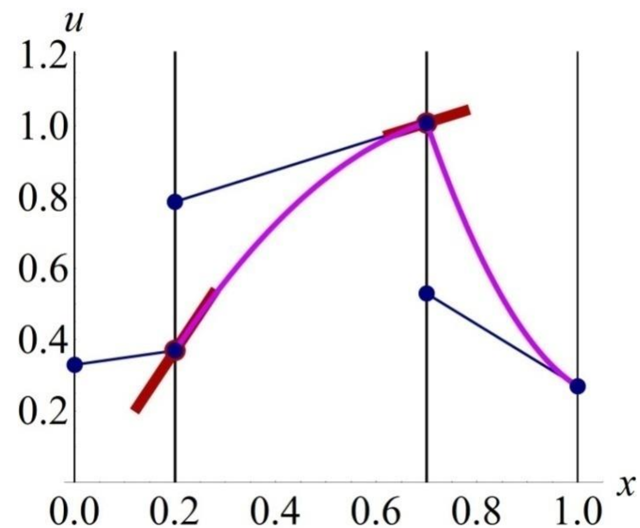
(a) Data



(b) DG



(c) Spectral Difference (SD)



(d) Scheme g_2

Correction Functions for Fourier-Stable Schemes

1. $$g_{\mathrm{DG}} = R_{R,k+1}$$

g_{DG} results in the DG method.

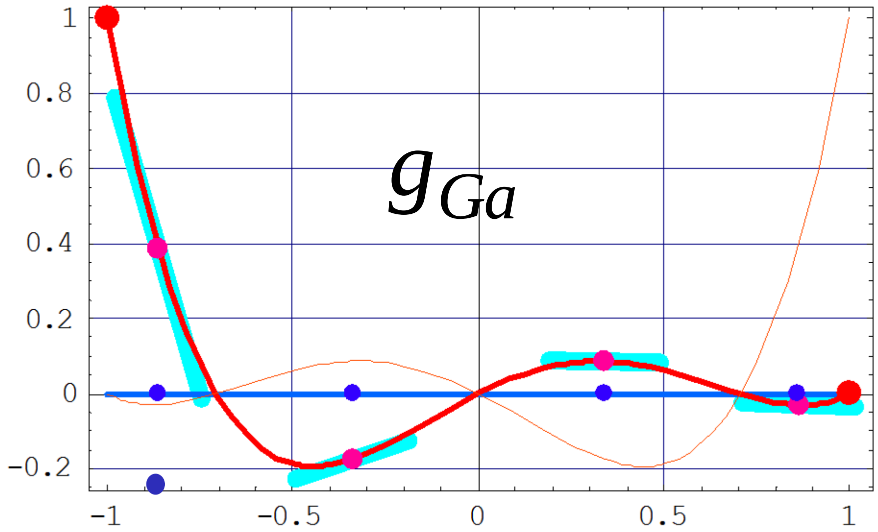
2. $$g_{Ga} = \frac{k+1}{2k+1} R_{R,k+1} + \frac{k}{2k+1} R_{R,k}$$

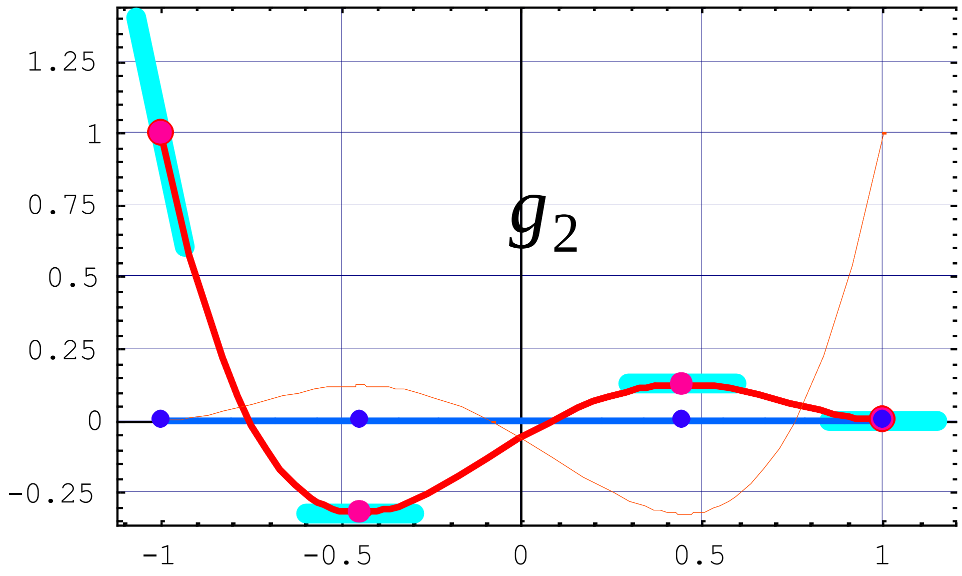
g_{Ga} vanishes at the k Gauss points

3. $$g_2 = \frac{k}{2k+1} R_{R,k+1} + \frac{k+1}{2k+1} R_{R,k}$$

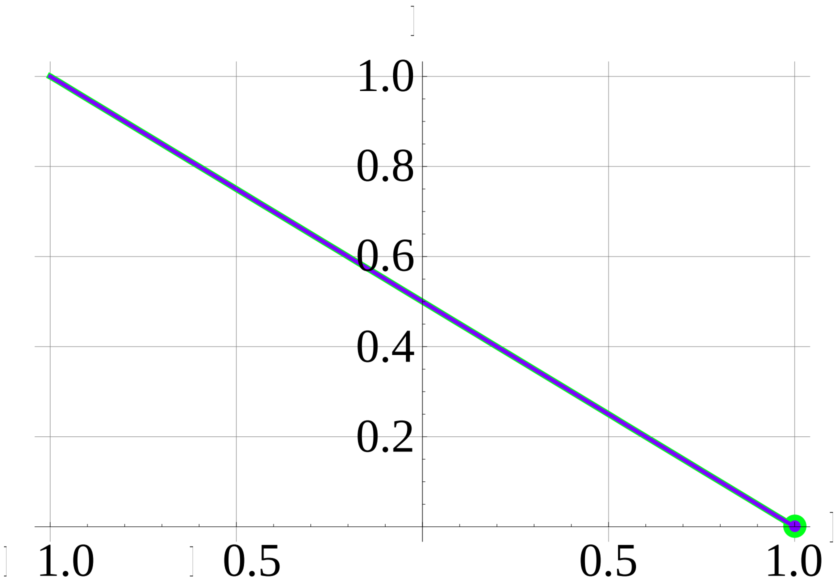
g_2' vanishes at k of the $k+1$ Lobatto points

Correction functions for $k = 3$

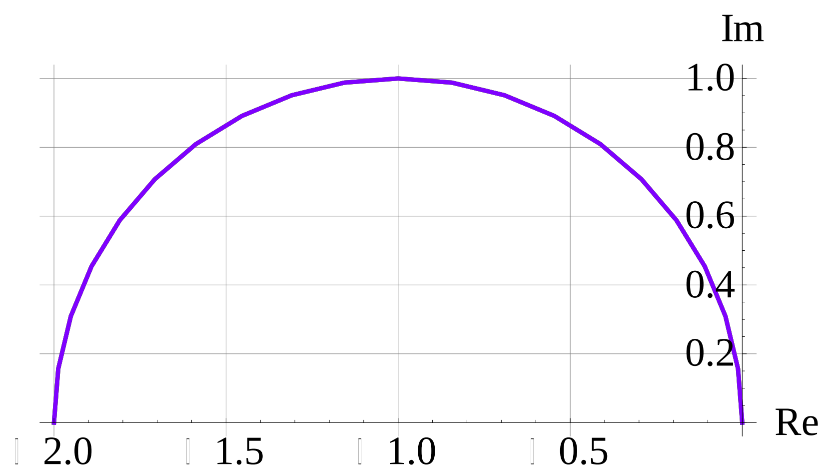




Fourier Analysis, $k = 0$



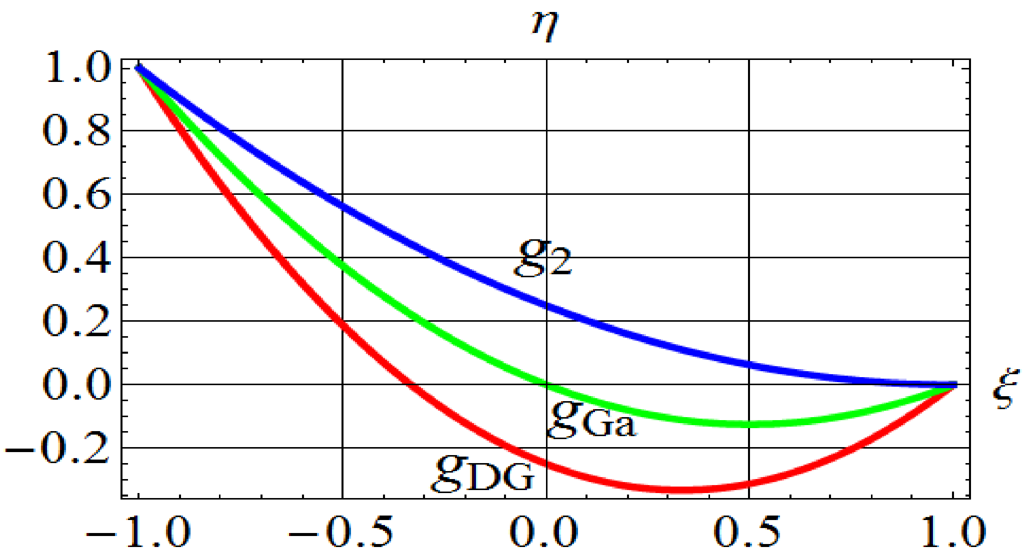
Correction Function



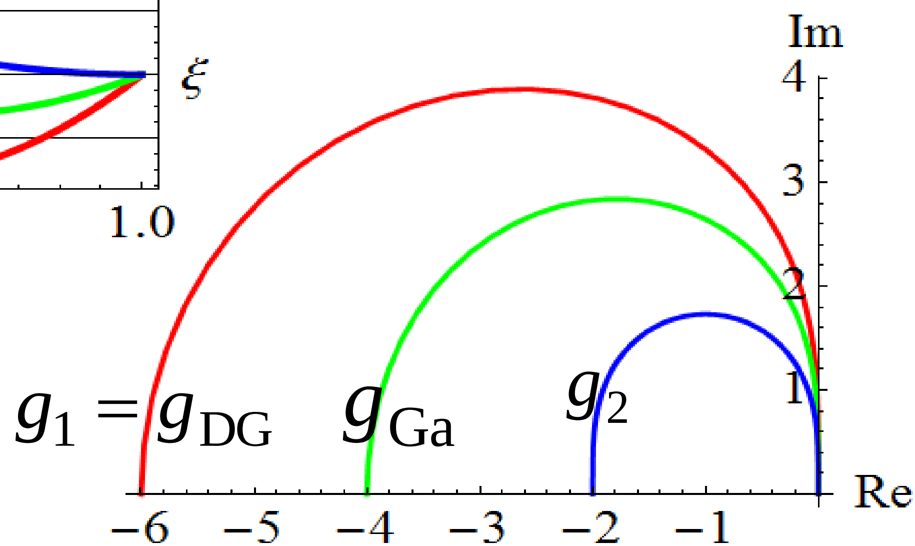
Spectra

Fourier Analysis, $k = 1$

$$g_{\mathrm{DG}} = \frac{3\xi^2}{4} - \frac{\xi}{2} - \frac{1}{4}, \quad g_{\mathrm{Ga}} = \frac{\xi^2}{2} - \frac{\xi}{2}, \quad \text{and} \quad g_2 = \frac{\xi^2}{4} - \frac{\xi}{2} + \frac{1}{4}.$$



Correction Functions



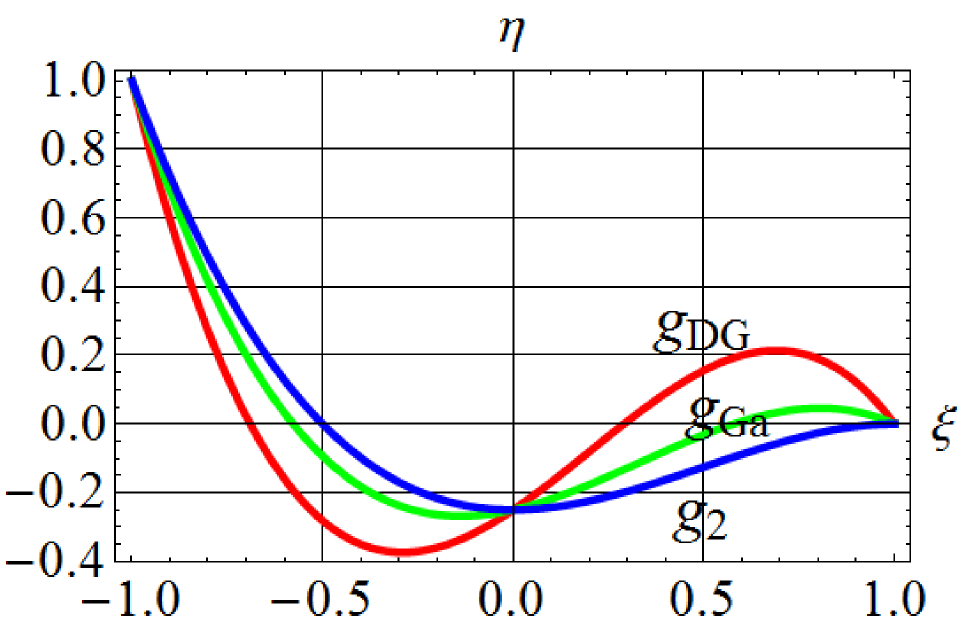
Spectra

Fourier Analysis, $k=1$

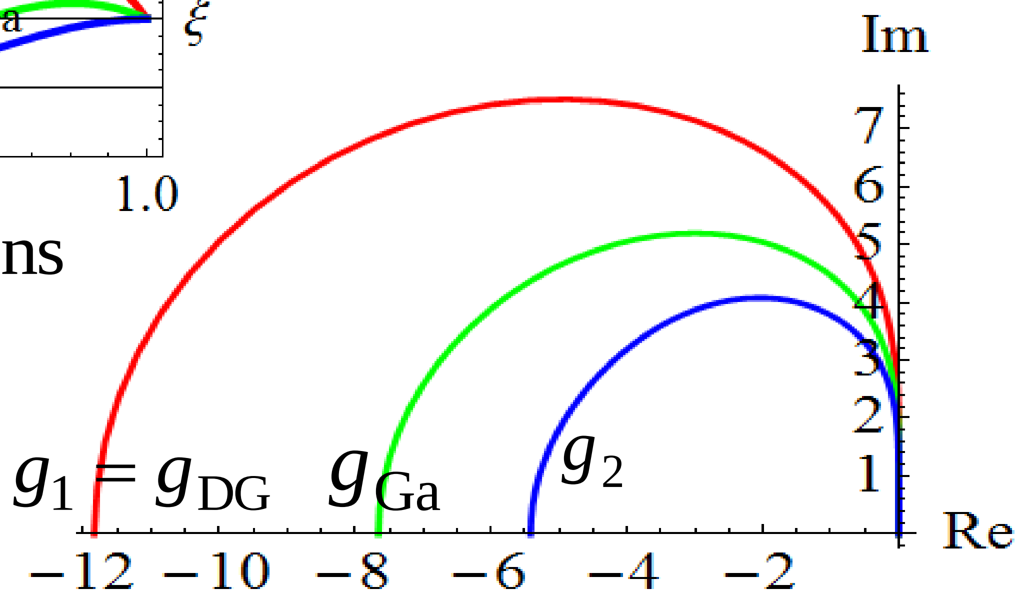
Orders of accuracy and errors

Scheme	Order of accuracy	Coarse mesh error, $w=\pi/8$	Fine mesh error, $w=\pi/16$
DG	3	$-3.2\times10^{-4}-3.3\times10^{-5}i$	$-2.1\times10^{-5}-1.1\times10^{-6}i$
g_{Ga}	2	$-7.1\times10^{-4}+2.4\times10^{-3}i$	$-4.6\times10^{-5}+3.1\times10^{-4}i$
g_2	2	$-2.5\times10^{-3}+9.\times10^{-3}i$	$-7.1\times10^{-4}+2.4\times10^{-3}i$

Fourier Analysis, $k = 2$



Correction Functions



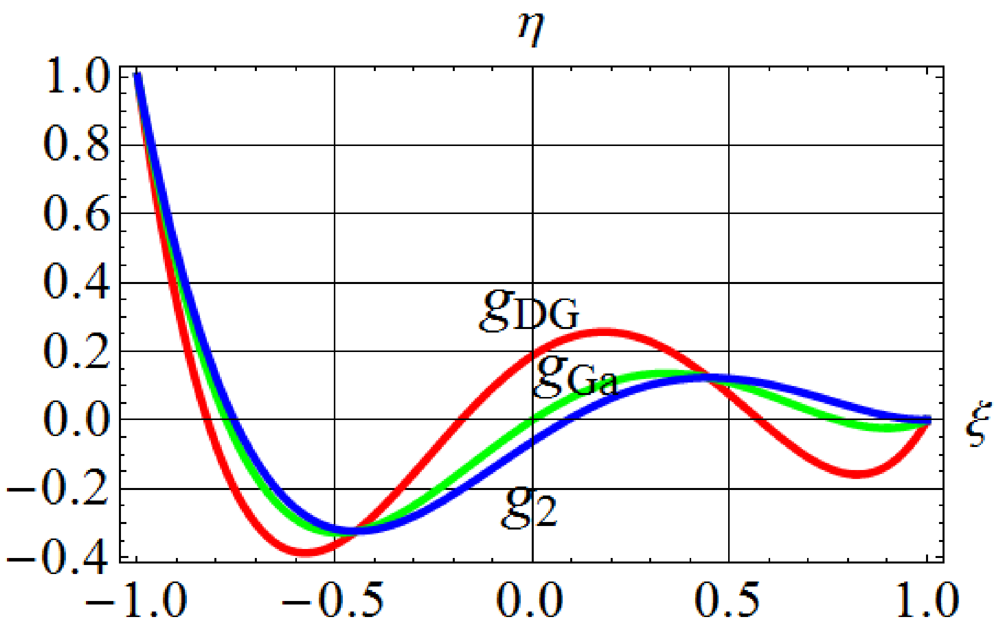
Spectra

Fourier Analysis, $k = 2$

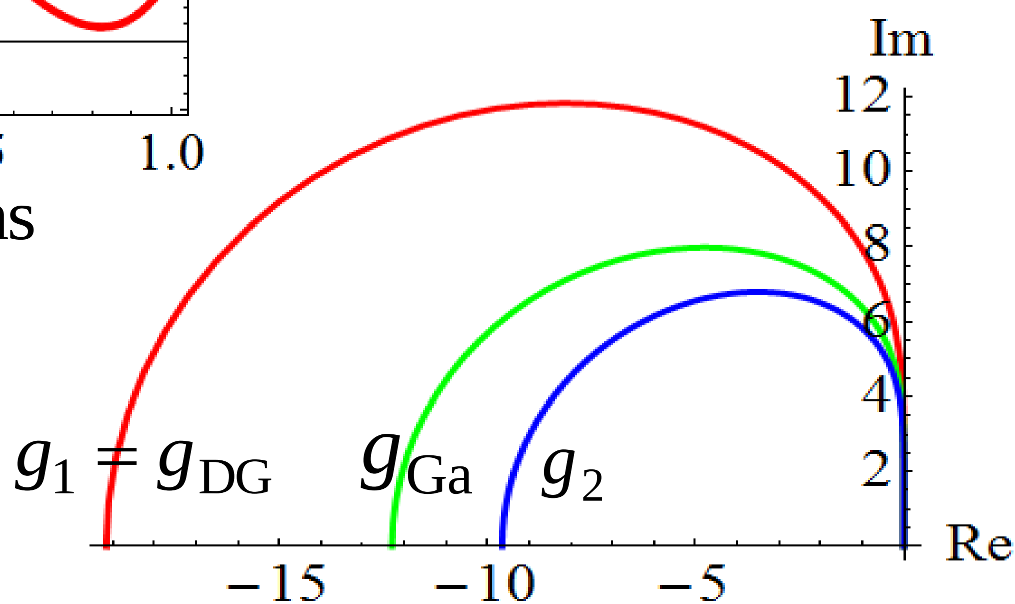
Orders of accuracy and errors

Scheme	Order of accuracy	Coarse mesh error, $w = \pi/8$	Fine mesh error, $w = \pi/16$
DG	5	$-5.\times 10^{-7} - 3.4\times 10^{-8} i$	$-7.9\times 10^{-9} - 2.7\times 10^{-10} i$
g_{Ga}	4	$-1.4\times 10^{-6} + 8.5\times 10^{-6} i$	$-2.2\times 10^{-8} + 2.7\times 10^{-7} i$
g_2	4	$-3.2\times 10^{-6} + 1.9\times 10^{-5} i$	$-5.\times 10^{-8} + 6.\times 10^{-7} i$

Fourier Analysis, $k = 3$



Correction Functions



Spectra

Fourier Analysis, $k = 3$

Orders of accuracy and errors

Scheme	Order of accuracy	Coarse mesh error, $w = \pi / 4$	Fine mesh error, $w = \pi / 8$
DG	7	$-1.\times 10^{-7} - 1.\times 10^{-8} i$	$-4.\times 10^{-10} - 2.\times 10^{-11} i$
g_{Ga}	6	$-3.1\times 10^{-7} + 1.3\times 10^{-6} i$	$-1.2\times 10^{-9} + 1.1\times 10^{-8} i$
g_2	6	$-5.4\times 10^{-7} + 2.3\times 10^{-6} i$	$-2.2\times 10^{-9} + 1.9\times 10^{-8} i$

Stability

$*$ For solutions of degee k, if g is orthogonal to $\boldsymbol{P}_{k-2}$,

then the (family) scheme is Fourier as well as energy-stable.

$*$ The above condition is not necessary: $g_{\text{Lump,Ch-Lo}}$ is not

orthogonal to any $\boldsymbol{P}_m$, but the resulting scheme is Fourier-stable.

Open problems

1. The collection of all g resulting in stable schemes remains to be identified.
2. Is Fourier stability equivalent to energy stability?

Energy Stability

- Jameson (2010) proved that a particular SD scheme (recovered via FR) is energy-stable.
- Vincent, Castonguay, and Jameson (2011) proved energy stability for a family of FR schemes.
- Energy-stability proofs for advection and advection diffusion equations in 1D, 2D, and 3D were provided by Vincent, Castonguay, Williams, and Jameson
- Can the current simplified proof for energy stability be extended to 2D, 3D, and tensor product cases?

Summary

- Review DG method
- New strong forms (approximate delta functions)
- Reconstruct the flux: FR methods
- Simplified energy-stability proof
- Open problems (for grad student, 1 month of study)
- NASA Report TM-2014-218135 June 1014 (pdf)
- There is significant current research activities in FR methods for practical flow problems in CFD.

Thank you for your attention.

Questions/Comments